SOME IDENTITIES ON BERNSTEIN POLYNOMIALS ASSOCIATED WITH q-EULER POLYNOMIALS

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Abstract In this paper we investigate some interesting properties of the q-Euler polynomials. The purpose of this paper is to give some relationships between Bernstein and q-Euler polynomials which are derived by the p-adic integral representation of the Bernstein polynomials associated with q-Euler polynomials.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic numbers and the field of p-adic completion of the algebraic closure of $\mathbb{Q}_p$, respectively (see [1–15]). Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The normalized p-adic absolute value is defined by $|p|_p = \frac{1}{p}$. As an indeterminate, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$\begin{aligned} I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-1}(x+p^N\mathbb{Z}_p) \qquad (1) \\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad \text{(see [7, 8, 9, 10])}. \end{aligned}$$

For $n \in \mathbb{N}$, we can derive the following integral equation from (1):

$$I_{-1}(f_n) = (-1)^n \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) + 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \qquad (2)$$

where $f_n(x) = f(x+n)$ (see [7, 8, 9, 10, 11]). As well known definition, the Euler polynomials are given by the generating function as follows:

$$\frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t}{n!}, \quad \text{(see [7, 8, 9, 10, 11, 12, 13, 5, 15, 14, 3])}, \quad (3)$$

with usual convention about replacing $E^n(x)$ by $E_n(x)$. In the special case $x = 0$, $E_n(0) = E_n$ are called the n-th Euler numbers. From (3), we can derive the following recurrence formula for Euler numbers

$$E_0 = 1, (E+1)^n + E_n = \begin{cases} 2 & \text{if } n = 0, \\ 0 & \text{if } n > 0, \quad \text{(see [12])}. \end{cases}$$

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with usual convention about replacing E^n by E_n. By the definitions of Euler numbers and polynomials, we get

$$E_n(x) = (E+x)^n = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_l, \quad \text{(see [7, 8, 9, 10, 11, 12, 13, 5, 15, 14, 3])}.$$

Let $C[0,1]$ denote the set of continuous functions on $[0,1]$. For $f \in C[0,1]$, Bernstein introduced the following well-known linear positive operator in the field of real numbers $\mathbb{R}$:

$$\mathbb{B}_n(f|x) = \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} f(\frac{k}{n}) B_{k,n}(x),$$

where $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$ (see [1, 2, 7, 11, 12, 14]). Here, $\mathbb{B}_n(f|x)$ is called the Bernstein operator of order n for f. For $k, n \in \mathbb{Z}_+$, the Bernstein polynomials of degree n are defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{for } x \in [0,1]. \tag{4}$$

In this paper, we study the properties of q-Euler numbers and polynomials. From these properties we investigate some identities on the q-Euler numbers and polynomials. Finally, we give some relationships between Bernstein and q-Euler polynomials, which are derived by the p-adic integral representation of the Bernstein polynomials associated with q-Euler polynomials.

2. q-Euler numbers and polynomials

In this ection we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. Let $f(x) = q^x e^{xt}$. From (1) and (2), we have

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \frac{2}{qe^t+1} e^{xt}. \tag{5}$$

Now, we define the q-Euler numbers as follows:

$$\frac{2}{qe^t+1} = e^{E_q t} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}, \tag{6}$$

with the usual convention about replacing E_q^n by $E_{n,q}$.

By (6), we easily get

$$E_{0,q} = \frac{2}{q+1}, \text{ and } q(E_q+1)^n + E_{n,q} = \begin{cases} 2 & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases} \tag{7}$$

with usual convention about replacing E_q^n by $E_{n,q}$.
We note that

$$\frac{2}{qe^t+1} = \frac{2}{e^t+q^{-1}} . \frac{2}{1+q} = \frac{2}{1+q} \sum_{n=0}^{\infty} H_n(-q^{-1}) \frac{t^n}{n!}, \tag{8}$$

where $H_n(-q^{-1})$ is the n-th Frobenius-Euler numbers.
From (5), (6) and (8), we have

$$\int_{\mathbb{Z}_p} q^x e^{xt} d\mu_{-1}(x) = E_{n,q} = \frac{2}{1+q} H_n(-q^{-1}), \text{ for } n \in \mathbb{Z}_+. \tag{9}$$

Now, we consider the q-Euler polynomials as follows:

$$\frac{2}{qe^t+1}e^{xt} = e^{E_q(x)t} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}, \tag{10}$$

whith the usual convention $E_q^n(x)$ by $E_{n,q}(x)$.

From (2), (5) and (10), we get

$$\int_{\mathbb{Z}_p} q^x e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{qe^t+1}e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}. \tag{11}$$

By comparing the coefficients on the both sides of (10) and (11), we get the following Witt's formula for the q-Euler polynomials as follows:

$$\int_{\mathbb{Z}_p} q^y (x+y)^n d\mu_{-1}(y) = E_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_{l,q}. \tag{12}$$

From (10) and (12), we can derive the following equation:

$$\frac{2q}{qe^t+1}e^{(1-x)t} = \frac{2}{1+q^{-1}e^{-t}}e^{-xt} = \sum_{n=0}^{\infty} E_{n,q^{-1}}(x)(-1)^n \frac{t}{n!}. \tag{13}$$

By (10) and (13), we obtain the following reflection symmetric property for the q-Euler polynomials.

Theorem 1. *For $n \in \mathbb{Z}_+$, we have*

$$(-1)^n E_{n,q^{-1}}(x) = qE_{n,q}(1-x).$$

From (9), (10), (11) and (12), we can derive the following equation: for $n \in \mathbb{N}$,

$$\begin{aligned}
E_{n,q}(2) &= (E_q+1+1)^n = \sum_{l=0}^{n} \binom{n}{l} E_{l,q}(1) \\
&= E_{0,q} + \frac{1}{q}\sum_{l=1}^{n} \binom{n}{l} qE_{l,q}(1) = \frac{2}{1+q} - \frac{1}{q}\sum_{l=1}^{n} \binom{n}{l} E_{l,q} \\
&= \frac{2}{1+q} + \frac{2}{q(1+q)} - \frac{1}{q}\sum_{l=0}^{n} \binom{n}{l} E_{l,q} \\
&= \frac{2}{q} - \frac{1}{q^2}qE_{n,q}(1) = \frac{2}{q} + \frac{1}{q^2}E_{n,q}, \quad \text{by using recurrence formula (7).}
\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 2. *For $n \in \mathbb{N}$, we have*

$$qE_{n,q}(2) = 2 + \frac{1}{q}E_{n,q}.$$

By using (9) and (12), we get

$$\begin{aligned}
\int_{\mathbb{Z}_p} q^{-x}(1-x)^n d\mu_{-1}(x) &= (-1)^n \int_{\mathbb{Z}_p} q^{-x}(x-1)^n d\mu_{-1}(x) \\
&= (-1)^n E_{n,q^{-1}}(-1) = q\int_{\mathbb{Z}_p} (x+2)^n d\mu_{-1}(x) = q\left(\frac{2}{q} + \frac{1}{q^2}E_{n,q}\right) \\
&= 2 + \frac{1}{q}E_{n,q} = 2 + \frac{1}{q}\int_{\mathbb{Z}_p} x^n q^x d\mu_{-1}(x), \text{ for } n > 0.
\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 3. *For $n \in \mathbb{N}$, we have*

$$\int_{\mathbb{Z}_p} q^{-x}(1-x)^n d\mu_{-1}(x) = 2 + \frac{1}{q}\int_{\mathbb{Z}_p} x^n q^x d\mu_{-1}(x).$$

By using Theorem 3, we will study for the p-adic integral representation on $\mathbb{Z}_p$ of the Bernstein polynomials associated with q-Euler polynomials in the next section.

3. Bernstein polynomials associated with q-Euler numbers and polynomials

Now, we take the p-adic integral on $\mathbb{Z}_p$ for the Bernstein polynomials in (4) as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}(x) q^x d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} \binom{n}{k} x^k (1-x)^{n-k} q^x d\mu_{-1}(x) \\ &= \binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^{n-k-j}\int_{\mathbb{Z}_p} x^{n-j} q^x d\mu_{-1}(x) \\ &= \binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^{n-k-j} E_{n-j,q} \\ &= \binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^{j} E_{k+j,q}, \quad \text{where } n,k \in \mathbb{Z}_+. \end{aligned} \tag{14}$$

By the definition of Bernstein polynomials, we see that

$$B_{k,n}(x) = B_{n-k,n}(1-x), \quad \text{where } n,k \in \mathbb{Z}_+. \tag{15}$$

Let $n,k \in \mathbb{Z}_+$ with $n > k$. Then, by (15), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} q^x B_{k,n}(x) d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} q^x B_{n-k,n}(1-x) d\mu_{-1}(x) \\ &= \binom{n}{n-k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\int_{\mathbb{Z}_p}(1-x)^{n-j} q^x d\mu_{-1}(x) \\ &= \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\left(2 + q\int_{\mathbb{Z}_p} x^{n-j} q^x d\mu_{-1}(x)\right) \\ &= \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\left(2 + qE_{n-j,q^{-1}}\right) \\ &= \begin{cases} 2 + qE_{n,q^{-1}} & \text{if } k = 0, \\ \binom{n}{k} q \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j} E_{n-j,q^{-1}} & \text{if } k > 0. \end{cases} \end{aligned}$$

Thus, we obtain the following theorem.

Theorem 4. *For $n, k \in \mathbb{Z}_+$ with $n > k$, we have*

$$\int_{\mathbb{Z}_p} q^{1-x} B_{k,n}(x) d\mu_{-1}(x) = \begin{cases} 2q + E_{n,q} & \text{if } k = 0, \\ \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} E_{n-j,q} & \text{if } k > 0. \end{cases}$$

By (14) and Theorem 4, we get the following corollary.

Corollary 5. *For $n, k \in \mathbb{Z}_+$ with $n > k$, we have*

$$\sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j E_{k+j,q^{-1}} = \begin{cases} 2 + \frac{1}{q} E_{n,q} & \text{if } k = 0, \\ \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \frac{1}{q} E_{n-j,q} & \text{if } k > 0. \end{cases}$$

For $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k$. Then we get

$$\begin{aligned} &\int_{\mathbb{Z}_p} B_{k,n}(x) B_{k,m}(x) q^{-x} d\mu_{-1}(x) \\ &= \binom{n}{k}\binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} \int_{\mathbb{Z}_p} q^{-x} (1-x)^{n+m-j} d\mu_{-1}(x) \\ &= \binom{n}{k}\binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} q \int_{\mathbb{Z}_p} (x+2)^{n+m-j} q^x d\mu_{-1}(x) \\ &= \binom{n}{k}\binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} q \left(\frac{2}{q} + \frac{1}{q^2} E_{n+m-j,q} \right) \\ &= \begin{cases} 2 + \frac{1}{q} E_{n+m,q} & \text{if } k = 0, \\ \binom{n}{k}\binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} \frac{1}{q} E_{n+m-j,q} & \text{if } k > 0. \end{cases} \end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 6. *For $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k$, we have*

$$\begin{aligned} &\int_{\mathbb{Z}_p} B_{k,n}(x) B_{k,m}(x) q^{1-x} d\mu_{-1}(x) \\ &= \begin{cases} 2q + E_{n+m,q} & \text{if } k = 0, \\ \binom{n}{k}\binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} E_{n+m-j,q} & \text{if } k > 0. \end{cases} \end{aligned}$$

By using binomial theorem, for $m, n, k \in \mathbb{Z}_+$, we get

$$\int_{\mathbb{Z}_p} B_{k,n}(x) B_{k,m}(x) q^{1-x} dd\mu_{-1}(x) \tag{16}$$

$$= \binom{n}{k}\binom{m}{k} \sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (-1)^j \int_{\mathbb{Z}_p} x^{j+2k} q^{1-x} d\mu_{-1}(x) \tag{17}$$

$$= q\binom{n}{k}\binom{m}{k} \sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (-1)^j E_{j+2k,q^{-1}}. \tag{18}$$

By comparing the coefficients on the both sides of (16) and Theorem 6, we obtain the following corollary.

Corollary 7. *Let $m, n, k \in \mathbb{Z}_+$ with $m+n > 2k$. Then we get*

$$\sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (-1)^j E_{j+2k,q^{-1}}$$
$$= \begin{cases} 2 + \frac{1}{q} E_{n+m,q} & \text{if } \ k = 0, \\ \frac{1}{q} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} E_{n+m-j,q} & \text{if } \ k > 0. \end{cases}$$

For $s \in \mathbb{N}$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk$. By induction, we get

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x) \cdots B_{k,n_s}(x) q^{-x} d\mu_{-1}(x) = \left(\prod_{i=1}^{s} \binom{n_i}{k}\right) \int_{\mathbb{Z}_p} x^{sk} (1-x)^{n_1+\ldots+n_s} q^{-x} d\mu_{-1}(x)$$

$$= \left(\prod_{i=1}^{s} \binom{n_i}{k}\right) \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} \int_{\mathbb{Z}_p} (1-x)^{n_1+\cdots+n_s-j} q^{-x} d\mu_{-1}(x)$$
$$= \left(\prod_{i=1}^{s} \binom{n_i}{k}\right) \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} q \int_{\mathbb{Z}_p} (x+2)^{n_1+\cdots+n_s-j} q^{x} d\mu_{-1}(x)$$
$$= \left(\prod_{i=1}^{s} \binom{n_i}{k}\right) \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} q \left(\frac{2}{q} + \frac{1}{q^2} E_{n_1+\cdots+n_s-j,q}\right)$$
$$= \begin{cases} 2 + \frac{1}{q} E_{n_1+\cdots+n_s,q} & \text{if } \ k = 0, \\ \left(\prod_{i=1}^{s} \binom{n_i}{k}\right) \frac{1}{q} \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} E_{n_1+\cdots+n_s-j,q} & \text{if } \ k > 0. \end{cases}$$

Therefore we obtain the following theorem.

Theorem 8. *Let $s \in \mathbb{N}$. For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk$, we have*

$$\int_{\mathbb{Z}_p} \left(\prod_{i=1}^{s} B_{k,n_i}(x) \right) q^{1-x} d\mu_{-1}(x)$$
$$= \begin{cases} 2q + E_{n_1+n_2+\cdots+n_s,q} & \text{if} \quad k = 0, \\ \left(\prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} E_{n_1+n_2+\cdots+n_s-j,q} & \text{if} \quad k > 0. \end{cases}$$

For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ by binomial theorem, we get

$$\int_{\mathbb{Z}_p} \left(\prod_{i=1}^{s} B_{k,n_i}(x) \right) q^{-x} d\mu_{-1}(x) \tag{19}$$
$$= \binom{n_1}{k} \ldots \binom{n_s}{k} \sum_{j=0}^{n_1+\cdots+n_s-sk} \binom{n_1 + \cdots + n_s - sk}{j} (-1)^j \int_{\mathbb{Z}_p} x^{j+sk} q^{-x} d\mu_{-1}(x)$$
$$= \binom{n_1}{k} \ldots \binom{n_s}{k} \sum_{j=0}^{n_1+\cdots+n_s-sk} \binom{n_1 + \cdots + n_s - sk}{j} (-1)^j E_{j+sk,q^{-1}}.$$

By using (19) and Theorem 8, we obtain the following corollary.

Corollary 9. *Let $s \in \mathbb{N}$. For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk$, we have*

$$\sum_{j=0}^{n_1+\cdots+n_s-sk} \binom{n_1 + \cdots + n_s - sk}{j} (-1)^j E_{j+sk,q^{-1}}$$
$$= \begin{cases} 2 + \frac{1}{q} E_{n_1+n_2+\cdots+n_s,q} & \text{if} \ \ k = 0, \\ \frac{1}{q} \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} E_{n_1+n_2+\cdots+n_s-j,q} & \text{if} \ \ k > 0. \end{cases}$$

References

[1] M. Acikgoz, S. Araci, *A study on the integral of the product of several type Berstein polynomials*, IST Transaction of Applied Mathematics-Modelling and Simulation, 2010, Vol. 1, No.1(2), ISBN 1913-8342, 10–14.

[2] S. Bernstein, *Démonstration du théorème de Weierstrass, fondée sur le calcul des probabilités*, Commun. Soc. Math. Kharkow **13** (1912), 1–2.

[3] I. N. Cangul, V. Kurt, H. Ozden, Y. Simsek, *On the higher-order w-q-Genocchi numbers*, Adv. Stud. Contemp. Math. **19** (2009), 39–57.

[4] N. K. Govil, V. Gupta, *Convergence of q-Meyer-König-Zeller-Durrmeyer operators*, Adv. Stud. Contemp. Math. **19** (2009), 97–108.

[5] L.C. Jang, W-J. Kim, Y. Simsek, *A study on the p-adic integral representation on $\mathbb{Z}_p$ associated with Bernstein and Bernoulli polynomials*, Advances in difference Equations, Vol 2010 (2010), Article ID 163219, 6pp.

[6] K. I Joy, *Bernstein polynomials*, On-line Geometric Modelling Notes, http://en.wikipedia.org/wiki/Bernstein polynomails.

[7] T. Kim, *Barnes-type multiple q-zeta functions and q-Euler polynomials*, J. Physics A : Math. Theor., **43** (2010), 255201, 11pp.

[8] T. Kim, J. Choi, Y. H.Kim, L. Jang, *On p-adic analogue of q-Bernstein polynomials and related integrals*, Discrete Dynamics in Nature and Society, Vol 2010 (2010), Article ID 17930, 3 pp.

[9] T. Kim, J. Choi, Y. H.Kim, *q-Bernstein polynomials associated with q-Stirling numbers and Carlit's q-Bernoulli numbers,* Abstract and Applied Analysis (in press).

[10] T. Kim, *Some identities on the q-Euler polynomials of higner order order and q-Stirling numbers by fermionic p-adic integral on $\mathbb{Z}_p$,* Russian Journal of Mathematical physics 16 (2009), 484–491.

[11] T. Kim, *Note on the Euler numbers and polynomials*, Adv. Stud. Contemp. Math. **19** (2008), 131–136.

[12] T. Kim, B. Lee, *Some identities of the Frobenius-Euler polynomials,* Abstract and Applied Analysis 2009 (2009), Article ID 639439,7 pp.

[13] T. Kim, *Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$,* Russian Journal of Mathematical physics 16 (2009), 484–491.

[14] V. Kurt,*A further symmetric relation on the analogue of the Apostol-Bernoulli and the analogue of the Apostol-Genocchi polynomials*, Appl. Math. Sci. **3** (2009), 53–56.

[15] S-H. Rim, J-H.Jin, E-J. Moon, S-J. Lee, *On multiple interpolation functions of the q-Genocchi polynomials,* Journal of inequalities and Applications, Vol 2010 (2010), Article ID 351419, 13 pp..

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